On the possibility of testing the Weak Equivalence Principle with artificial Earth satellites

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Abstract

In this paper we examine the possibility of testing the equivalence principle, in its weak form, by analyzing the orbital motion of a pair of artificial satellites of different composition moving along orbits of identical shape and size in the gravitational field of Earth. It turns out that the obtainable level of accuracy is, realistically, of the order of 10^{-10} or slightly better. It is limited mainly by the fact that, due to the unavoidable orbital injection errors, it would not be possible to insert the satellites in orbits with exactly the same radius and that such difference could be known only with a finite precision. The present–day level of accuracy, obtained with torsion balance Earth–based measurements and the analysis of Earth–Moon motion in the gravitational field of Sun with the Lunar Laser Ranging technique, is of the order of 10^{-13}. The proposed space–based missions STEP, μSCOPE, GG and SEE aim to reach a 10^{-15}–10^{-18} precision level.

1 Introduction

The Weak Equivalence Principle (WEP) and the Einstein Equivalence Prnciple (EEP) (Will 1993; Ciufolini and Wheeler 1995; Haugan and Lämmerzahl 2001; Will 2001) are the cornerstones of Einstein General Theory of Relativity (GTR) and of all the other competing metric theories of gravity. The WEP states that non–rotating, uncharged bodies of different structure and compositions and with negligible amount of gravitational binding energy per unit mass fall with the same acceleration in a given gravitational field, provided that no other forces act on them. The EEP (called also Medium Strong Equivalence Principle) states that the outcome of any local, non–gravitational test experiment is independent of where and when in the gravitational field the experiment is performed. If the gravitational binding energy per unit mass of the freely falling bodies in a given external gravitational field is not negligible, as is the case for astronomical bodies, we have the Strong Equivalence Priciple (SEP. It is called also Very Strong Equivalence Principle). When the self–gravity of the falling bodies is accounted for it might happen, in principle, that their accelerations are different. This violation of the universality of free fall, called Nordvedt effect (Nordvedt 1968a), is predicted by all metric theories of gravity apart from GTR which, instead, satisfies the SEP. Lunar Laser Ranging (LLR) is able to test this effect for the motion of Earth and Moon in the gravitational field of Sun (Nordvedt 1968b). No different accelerations have been found at a 10^{-13} level (Anderson and Williams 2001). Of course, this is also a test of the WEP.

To go to space, where much larger distances, velocities, gravitational potential differences with respect to Earth, and, most important, free fall for an, in principle, infinitely long time are available, is important to perform very accurate tests of post–Newtonian gravity. In this paper we wish to investigate the level of accuracy which could be reached in testing the WEP, by analysing the orbital motion of a pair of Earth artificial satellites of different compositions for which the effects of self–gravity would be, of course, negligible: it is analogous to the analysis of the motion of Earth and Moon in the gravitational field of Sun in order to test the SEP. For the already performed experimental tests of the WEP on Earth see (Will 2001) and the references therein. The present–day level of accuracy is 4×10^{-13} (Bässler *et al* 1999). It has been obtained in the experiments of the so–called Eöt-Wash group of the Washington University by means of

a sophisticated torsion balance in an Earth based laboratory set–up. Other proposed space–based experiments are the very complex and expensive STEP (Lockerbie *et al* 2001), μSCOPE (Touboul 2001), GG (Nobili *et al* 2000) and SEE (Sanders *et al* 2000) missions[1] whose goal is to reach the 10^{-15}–10^{-18} level of accuracy.

The reason for searching for more and more accurate tests of the equivalence principle resides in the fact that all approaches to quantizing gravity and to unifying it with the other fundamental interactions currently under study are capable of predicting violations of the equivalence principle at some level. For example, departures from universal free fall accelerations of the order of 10^{-15} have been calculated in (Damour and Polyakov 1994a; 1994b) in the string theory context. Violations of the WEP are predicted also by nonsymmetric theories of gravity (Will 1989). In (Moffat and Gillies 2002) a violation of the WEP of the order of 10^{-14} is predicted in the context of a non–local quantum gravity theory.

2 The orbital period

In this section we examine the possibility of testing the WEP by measuring the orbital periods of a pair of Earth artificial satellites of different compositions. It can be thought of as a comparison between two pendulums with enormously long threads swinging for an extremely long time[2].

The orbital period of a satellite freely falling in Earth's gravitational field can be written as

$$T = \frac{2\pi}{n(1+\Delta n)} \sim \frac{2\pi}{n}(1-\Delta n), \tag{1}$$

in which the Keplerian unperturbed period is

$$T^{(0)} = \frac{2\pi}{n} = 2\pi\sqrt{\frac{a^3}{GM}} \times \sqrt{\frac{m_{\rm i}}{m_{\rm g}}} = 2\pi\sqrt{\frac{a^3}{GM}} \times \psi, \tag{2}$$

[1] See also on the WEB http://einstein.stanford.edu/STEP/, http://www.onera.fr/microscope/ http://tycho.dm.unipi.it/∼nobili/ggproject.html, and http://www.phys.utk.edu/see/. Notice that while the first three missions are in advanced stages of planning and hardware testing, and are expected to be launched in the next few years, SEE is still undergoing rigorous conceptual evaluation and is not yet a scheduled mission.

[2] In the case of the SEP–LLR experiment the amplitude of the parallactic inequality long–period harmonic perturbation, proportional to $\cos D$, where D is the synodic phase from New Moon, is sensitive to the possible different falling rates of Earth and Moon toward the Sun.

where G, M and a are the Newtonian gravitational constant, the mass of Earth and the satellite semimajor axis, respectively. We have explicitly written the square root of the ratio of the inertial to the (passive) gravitational mass of satellite ψ. The quantity Δn represents the various kind of perturbations, of gravitational and non–gravitational origin, which affect n. For example, the even zonal harmonic coefficients of the multipolar expansion of Earth gravitational potential, called geopotential, induce secular perturbations on n. The most important one is that due to the first even zonal harmonic J_2 and, for a circular orbit with eccentricity $e=0$, it is given by

$$\Delta n_{\rm obl}^{(\ell=2)} = -\frac{3}{4}J_2\left(\frac{R}{a}\right)^2(1-3\cos^2 i), \tag{3}$$

where R is the Earth radius and i is the inclination of the orbital plane to the equator. Also the time–varying part of Earth gravitational field should be considered, in principle, because the Earth solid and ocean tides (Iorio 2001a) induce long–period harmonic perturbations on n. For a given tidal line of frequency f the perturbations induced by the solid tides, which are the most effective in affecting the orbits of a satellite, can be written as

$$\Delta n^{\ell}_{\rm tides} = \sum_{m=0}^{\ell}\left(\frac{H_{\ell}^{m}}{R}\right)\left(\frac{R}{a}\right)^{\ell}k_{\ell m}^{(0)}A_{\ell m}\sum_{p=0}^{\ell}\sum_{q=-\infty}^{+\infty}F_{\ell mp}\left[2(\ell+1)G_{\ell pq}-\frac{(1-e^2)}{e}\frac{dG_{\ell pq}}{de}\right]\cos\gamma_{f\ell mpq}, \tag{4}$$

where H_{ℓ}^{m} are the tidal heights, $k_{\ell m}^{(0)}$ are the Love numbers, $A_{\ell m}=\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}$, $F_{\ell mp}(i)$ and $G_{\ell pq}(e)$ are the inclination and the eccentricity functions (Kaula 1966), respectively, and $\gamma_{f\ell mpq}$ is the frequency of the tidal perturbation built up with the frequencies of the lunisolar variables and of the satellite's orbital elements.

In order to test the WEP, we propose to measure, after many revolutions, the difference of time spans which are multiple N of the orbital periods $\Delta T_N \equiv T_N^{(2)} - T_N^{(1)}$ of a couple of satellites of different composition orbiting the Earth along circular orbits of almost same radius a

$$a^{(1)} \equiv a,\ a^{(2)} = a + d. \tag{5}$$

The small difference d, which, in principle, should be equal to zero, is due to the unavoidable orbital injection errors. It can be made extremely small with a rocket launcher of good quality: for example, at the beginning of their mission the semimajor axes of the two GRACE spacecrafts

were different by an amount of just[3] 0.5 km.

The observable quantity in which we are interested is $\Delta\psi$ to be measured after N orbital revolutions, where

$$\psi^{(1)} \equiv \psi = 1,\ \psi^{(2)} = \psi + \Delta\psi, \tag{6}$$

with $\Delta\psi/\psi \ll 1$. The quantity $\Delta\psi$ can be expressed in terms of the standard Eötvös parameter η. Indeed, the inertial mass of a body is composed by many types of mass–energy: rest energy, electromagnetic energy, weak–interaction energy, and so on. If one of these forms of energy contributes to $m_{\rm g}$ differently than it does to $m_{\rm i}$ we can put (Will 1993)

$$m_{\rm g} = m_{\rm i} + \sum_A \eta^A \frac{E^A}{c^2}, \tag{7}$$

where E^A is the internal energy of the body generated by the interaction A and η^A is a dimensionless parameter that measures the strength of the violation of WEP induced by that interaction. Then

$$\frac{m_{\rm i}^{(2)}}{m_{\rm g}^{(2)}} - \frac{m_{\rm i}^{(1)}}{m_{\rm g}^{(1)}} \sim \sum_A \eta^A \left[\frac{E^A_{(1)}}{m_{\rm i}^{(1)}c^2} - \frac{E^A_{(2)}}{m_{\rm i}^{(2)}c^2}\right] \equiv \eta. \tag{8}$$

From eq.(8) it can be obtained

$$\psi^{(2)} \equiv \sqrt{\frac{m_{\rm i}^{(2)}}{m_{\rm g}^{(2)}}} = \sqrt{\frac{m_{\rm i}^{(1)}}{m_{\rm g}^{(1)}} + \left[\frac{m_{\rm i}^{(2)}}{m_{\rm g}^{(2)}} - \frac{m_{\rm i}^{(1)}}{m_{\rm g}^{(1)}}\right]} = \sqrt{\frac{m_{\rm i}^{(1)}}{m_{\rm g}^{(1)}} + \eta} \sim \psi^{(1)} + \frac{\eta}{2\psi^{(1)}}, \tag{9}$$

so that, from eq.(6),

$$\Delta\psi = \frac{\eta}{2}. \tag{10}$$

From eqs.(1)-(2) we can write

$$\Delta T_N = N2\pi \left[\frac{1}{n^{(2)}} - \frac{1}{n^{(1)}} + \frac{\Delta n^{(1)}_{\rm obl}}{n^{(1)}} - \frac{\Delta n^{(2)}_{\rm obl}}{n^{(2)}}\right]. \tag{11}$$

In eq.(11) we consider only the gravitational even zonal perturbations due to Earth oblateness, as usually done in orbital reduction programs in which a reference orbit including the J_2 effects is adopted[4]. They can be summarized as

$$\Delta n_{\rm obl} = \sum_{\ell=2} \left(\frac{R}{a}\right)^{\ell} \mathcal{G}_{\ell}, \tag{12}$$

[3]See on the WEB http://www.csr.utexas.edu/grace/newsletter/2002/august2002.html
[4]This approximation will be justified later.

where the $\mathcal{G}_\ell = \mathcal{G}_\ell(i,\ e;\ J_\ell)$ functions include the even zonal harmonics coefficients J_ℓ, the eccentricity e, the inclination angle i and some numerical constants. For example, for $\ell = 2$ and $e = 0$, $\mathcal{G}_2 = -\frac{3}{4}J_2(1-3\cos^2 i)$. By using the expansions of eqs.(5)-(6) it is possible to solve eq.(11) for[5] $\Delta\psi_N$

$$\Delta\psi_N = \frac{\frac{\sqrt{GM}}{N\pi}\Delta T_N - (A+B)}{(A+C)+(B+D)}, \tag{13}$$

with

$$\begin{aligned} A &= 3d\sqrt{a}, & (14)\\ B &= \sum_{\ell=2} \mathcal{G}_\ell R^\ell \left[3\ell d^2 a^{-\left(\frac{1+2\ell}{2}\right)} + 2\ell d a^{\left(\frac{1-2\ell}{2}\right)} - 3da^{\left(\frac{1-2\ell}{2}\right)}\right], & (15)\\ C &= 2\sqrt{a^3}, & (16)\\ D &= \sum_{\ell=2} \mathcal{G}_\ell R^\ell \left[-2a^{\left(\frac{3-2\ell}{2}\right)}\right]. & (17) \end{aligned}$$

Eq.(13), together with eqs.(14)-(17), allows to evaluate the accuracy obtainable in measuring the quantity $\Delta\psi_N$. The error in measuring the difference of multiples of the orbital periods yields

$$[\delta(\Delta\psi_N)]_{\Delta T_N} = \frac{\sqrt{GM}}{N\pi}\frac{\delta(\Delta T_N)}{(A+C)+(B+D)}, \tag{18}$$

while the error in Earth GM, which amounts to 8×10^{11} cm^3 s^{-2} (McCarthy 1996), yields

$$[\delta(\Delta\psi_N)]_{GM} = \frac{(\Delta T_N)_{\rm exp}}{N2\pi\sqrt{GM}}\frac{\delta(GM)}{(A+C)+(B+D)}. \tag{19}$$

The uncertainty in Earth even zonal harmonics J_ℓ, of which δJ_2 amounts to 7.9626×10^{-11} (Lemoine *et al* 1998), has an impact given by

$$[\delta(\Delta\psi_N)]_{J_\ell} \leq \sum_{\ell=2}\left|\left\{\frac{-\frac{\partial B}{\partial J_\ell}\times[(A+C)+(B+D)] + (A+B)\times\frac{\partial(B+D)}{\partial J_\ell}}{[(A+C)+(B+D)]^2}\right\}\right| \times \delta J_\ell. \tag{20}$$

The errors in $\Delta\psi_N$ due to the uncertainties in a and in d are

$$[\delta(\Delta\psi_N)]_a = \left|\left\{\frac{-\frac{\partial(A+B)}{\partial a}\times[(A+C)+(B+D)] + (A+B)\times\frac{\partial[(A+C)+(B+D)]}{\partial a}}{[(A+C)+(B+D)]^2}\right\}\right| \times \delta a, \tag{21}$$

[5] Here we have considered $i^{(1)} = i^{(2)}$. Moreover, notice that if the two satellites would be inserted in counter–rotating orbits, in ΔT_N it must be included also the time shift $\Delta T_N^{\rm gm} \propto N\times 4\pi\frac{J}{c^2 M}$ due to the general relativistic gravitomagnetic clock effect (Mashhoon *et al* 1999; 2001; Iorio *et al* 2002) induced by the off-diagonal components of the metric proportional to the proper angular momentum J of Earth. Indeed, it turns out (Iorio *et al* 2002) that such time shift is independent of the ratio of the inertial to the gravitational masses of the satellite. It would act as a lower limit of the order of $N\times10^{-7}$ s to $(\Delta T_N)_{\rm exp}$ if it was detectable.

$$[\delta(\Delta\psi_N)]_d \;=\; \left|\left\{\frac{-\frac{\partial(A+B)}{\partial d}\times(C+D)}{[(A+C)+(B+D)]^2}\right\}\right|\times\delta d. \quad (22)$$

Notice that $[\delta(\Delta\psi_N)]_{J_\ell}$, $[\delta(\Delta\psi_N)]_a$ and $[\delta(\Delta\psi_N)]_d$ do not depend explicitly on the number of orbital revolutions N. In order to calculate them we need the explicit expressions of the derivatives with respect to J_ℓ

$$\frac{\partial B}{\partial J_\ell} = \left(\frac{\partial \mathcal{G}_\ell}{\partial J_\ell}\right) R^\ell \left[3\ell d^2 a^{-\left(\frac{1+2\ell}{2}\right)} + 2\ell d a^{\left(\frac{1-2\ell}{2}\right)} - 3da^{\left(\frac{1-2\ell}{2}\right)}\right], \quad (23)$$

$$\frac{\partial D}{\partial J_\ell} = \left(\frac{\partial \mathcal{G}_\ell}{\partial J_\ell}\right) R^\ell \left[-2a^{\left(\frac{3-2\ell}{2}\right)}\right], \quad (24)$$

with respect to a

$$\frac{\partial A}{\partial a} = \frac{3d}{2\sqrt{a}}, \quad (25)$$

$$\frac{\partial B}{\partial a} = \sum_{\ell=2}\mathcal{G}_\ell R^\ell\left[-3\frac{(1+2\ell)}{2}\ell d^2 a^{-\left(\frac{3+2\ell}{2}\right)} + (1-2\ell)\ell d a^{-\left(\frac{1+2\ell}{2}\right)} - 3\frac{(1-2\ell)}{2} d a^{-\left(\frac{1+2\ell}{2}\right)}\right], \quad (26)$$

$$\frac{\partial C}{\partial a} = 3\sqrt{a}, \quad (27)$$

$$\frac{\partial D}{\partial a} = \sum_{\ell=2}\mathcal{G}_\ell R^\ell\left[-(3-2\ell)a^{\left(\frac{1-2\ell}{2}\right)}\right], \quad (28)$$

and those with respect to d

$$\frac{\partial A}{\partial d} = 3\sqrt{a}, \quad (29)$$

$$\frac{\partial B}{\partial d} = \sum_{\ell=2}\mathcal{G}_\ell R^\ell\left[6\ell d a^{-\left(\frac{1+2\ell}{2}\right)} + 2\ell a^{\left(\frac{1-2\ell}{2}\right)} - 3a^{\left(\frac{1-2\ell}{2}\right)}\right]. \quad (30)$$

In order to fix the ideas, let us consider the orbit of the proposed LARES laser–ranged satellite with $a = 12270$ km, i=70 deg. Let us assume (Peterson 1997) $d = 5$ km. With these data we have, for $\ell = 2$,

$$[\delta(\Delta\psi_N)]_{\Delta T_N} = 7.3\times10^{-5}\ \mathrm{s}^{-1}\times\frac{\delta(\Delta T_N)}{N}, \quad (31)$$

$$[\delta(\Delta\psi_N)]_{GM} = 7.4\times10^{-14}\ \mathrm{s}^{-1}\times\frac{(\Delta T_N)_{\mathrm{exp}}}{N}, \quad (32)$$

$$[\delta(\Delta\psi_N)]_{J_2} = 8\times10^{-15}, \quad (33)$$

$$[\delta(\Delta\psi_N)]_a = 5\times10^{-13}\ \mathrm{cm}^{-1}\times\delta a, \quad (34)$$

$$[\delta(\Delta\psi_N)]_d = 1\times10^{-9}\ \mathrm{cm}^{-1}\times\delta d. \quad (35)$$

The accuracy in measuring the difference of the multiples of the orbital periods of the satellites is a crucial factor in obtaining a high precision in $\Delta\psi$. However, it could be possible, in principle, to choose an observational time span covering a very high number of orbital revolutions. It should be noted that $\delta(\Delta T_N)$ accounts for both the measurement errors and the systematical errors induced by various gravitational and non–gravitational aliasing phenomena. The latter ones play a very important role in strongly limiting the possibility of measuring, e.g., the gravitomagnetic clock effect: it amounts to 10^{-7} s after one orbital revolution for orbits with $e = i = 0$ while in (Iorio 2001b) it turned out that the systematic errors induced by the present–day level of knowledge of the terrestrial gravitational field are up to 2–3 order of magnitude larger. So, it should not be unrealistic to consider $\delta(\Delta T_N)_{\rm sys} \sim 10^{-4} - 10^{-5}$ s. This would imply $[\delta(\Delta\psi_N)]_{\Delta T_N} \sim \frac{10^{-9}-10^{-10}}{N}$.

The error due to the difference d in the semimajor axes of the satellites, which turns out to be the major limiting factor, cannot be reduced, in principle, by waiting for a sufficiently high number of orbital revolutions because it is independent of N. A small improvement could be obtained with the use of a larger semimajor axis. A geostationary orbit with $a = 42160$ km would allow to get $[\delta(\Delta\psi_N)]_{\Delta T_N} = 1 \times 10^{-5}\ {\rm s}^{-1} \times \frac{\delta(\Delta T_N)}{N}$ and $[\delta(\Delta\psi_N)]_d = 3 \times 10^{-10}\ {\rm cm}^{-1} \times \delta d$. However, in this case, for a fixed time span, we would have at our disposal a smaller N. In regard to the systematic part of the error δd, it should be noted that there are no secular or long–period perturbations of gravitational origin on the semimajor axis of a satellite. The non–gravitational perturbations could be reduced to a good level by adopting the drag–free technology. In regard to $\delta d_{\rm exp}$, it is important to note that in the present GRACE mission (Davis *et al* 1999), making use of a K/K$_{\rm a}$-band intersatellite link that provides dual one–way range measurements, changes in the distance of the two spacecrafts can be established with an accuracy of about 10^{-2} cm or even better.

It is interesting to note that if we neglect in all calculations B and D and the related derivatives, i.e., if we neglect the effects of Earth oblateness, it turns out that the numerical results of eqs.(31)-(35) do not change. It is very important because it means that our choice of neglecting the contribution of the non–gravitational perturbations in Δn is *a posteriori* correct. Indeed, the perturbing acceleration due to Earth J_2 on LAGEOS is of the order of 10^{-1} cm s^{-2}, while the impact of the direct solar radiation pressure, which is the largest non–gravitational

perturbation on LAGEOS, amounts to 10^{-7} cm s^{-2} (Milani *et al* 1987). Moreover, these conclusions imply that the errors in the inclination i, which enters the even zonal harmonic perturbations due to the geopotential and the non–gravitational perturbations, can be safely neglected, as done here. The same considerations hold also for the tidal perturbations: suffices it to say that the effects of the 18.6–year and the K_1 tides, which are the most powerful in perturbing the satellite orbits, on a LAGEOS–type satellite are six orders of magnitude smaller than those due to the static J_2 even zonal part of geopotential. Also the tiny general relativistic gravitoelectric correction to the orbital period induced by the Schwarzschild part of the metric[6], which depends on $\sqrt{m_{\rm i}/m_{\rm g}}$, can be neglected because for a LAGEOS–type satellite the disturbing acceleration is of the order of 9×10^{-8} cm s^{-2} (Milani *et al* 1987).

3 The longitude of the ascending node

The longitude of the ascending node Ω is one of the best accurately measured Keplerian orbital elements of Earth artificial satellites. Then, we wish to examine if it would be possible to use it in order to test the equivalence principle.

Let us recall that there are two kinds of long–period perturbations on the node Ω of an Earth satellite. First, the static oblateness of Earth induces a secular precession of Ω through the even zonal harmonics of the geopotential. Second, the time–varying part of Earth gravitational potential induces tidal harmonic perturbations on Ω (Iorio 2001a). Then, we can pose, by including the solid Earth tidal perturbations

$$\dot{\Omega} = n\sum_{\ell=2}\left(\frac{R}{a}\right)^{\ell}\left[\mathcal{G}_{\ell}+\sum_{m=0}^{\ell}\left(\frac{H_{\ell}^{m}}{R}\right)k_{\ell m}^{(0)}A_{\ell m}\sum_{p=0}^{\ell}\sum_{q=-\infty}^{+\infty}\frac{dF_{\ell mp}}{di}\frac{G_{\ell pq}}{\sin i\sqrt{1-e^{2}}}\cos\gamma_{f\ell mpq}\right], \qquad (36)$$

where, for $\ell=2$ and $e=0$, $\mathcal{G}_2=-\frac{3}{2}J_2\cos i$. By considering a couple of satellites of different compositions freely orbiting along almost identical orbits we could measure the difference of their secular nodal rates $\Delta\dot{\Omega}\equiv\dot{\Omega}^{(2)}-\dot{\Omega}^{(1)}$. In this case, by posing $\xi\equiv\sqrt{\frac{m_{\rm g}}{m_{\rm i}}}$, the violating parameter $\Delta\xi=-\frac{\eta}{2}$ can be expressed as

$$\Delta\xi=\frac{\frac{\Delta\dot{\Omega}}{\sqrt{GM}}+\mathcal{A}+\mathcal{B}}{\mathcal{C}-\mathcal{B}}, \qquad (37)$$

[6]For a circular orbit it is given by $T^{(0)}\times\Theta_{\rm ge}=\psi\frac{3\pi\sqrt{GMa}}{c^2}$ (Mashhoon *et al* 2001).

with[7]

$$\mathcal{A} = \sum_{\ell=2} R^{\ell}\Delta\mathcal{G}_{\ell} a^{-\left(\frac{3+2\ell}{2}\right)}, \quad (38)$$

$$\mathcal{B} = d\sum_{\ell=2} R^{\ell}\mathcal{G}_{\ell}\left(\frac{3+2\ell}{2}\right) a^{-\left(\frac{5+2\ell}{2}\right)}, \quad (39)$$

$$\mathcal{C} = \sum_{\ell=2} R^{\ell}\mathcal{G}_{\ell} a^{-\left(\frac{3+2\ell}{2}\right)}, \quad (40)$$

$$(41)$$

where $\Delta\mathcal{G}_{\ell} = \mathcal{G}_{\ell}^{(1)} - \mathcal{G}_{\ell}^{(2)}$ is the difference in the $\mathcal{G}_{\ell}$ functions of the two satellites induced by the inclinations and the eccentricities. The derivatives with respect to J_{ℓ}, a and d are

$$\frac{\partial\mathcal{A}}{\partial J_{\ell}} = \sum_{\ell=2} R^{\ell}\left(\frac{\partial\Delta\mathcal{G}_{\ell}}{\partial J_{\ell}}\right) a^{-\left(\frac{3+2\ell}{2}\right)}, \quad (42)$$

$$\frac{\partial\mathcal{B}}{\partial J_{\ell}} = d\sum_{\ell=2} R^{\ell}\left(\frac{\partial\mathcal{G}_{\ell}}{\partial J_{\ell}}\right)\left(\frac{3+2\ell}{2}\right) a^{-\left(\frac{5+2\ell}{2}\right)}, \quad (43)$$

$$\frac{\partial\mathcal{C}}{\partial J_{\ell}} = \sum_{\ell=2} R^{\ell}\left(\frac{\partial\mathcal{G}_{\ell}}{\partial J_{\ell}}\right) a^{-\left(\frac{3+2\ell}{2}\right)}, \quad (44)$$

$$\frac{\partial\mathcal{A}}{\partial a} = -\sum_{\ell=2} R^{\ell}\Delta\mathcal{G}_{\ell}\left(\frac{3+2\ell}{2}\right) a^{-\left(\frac{5+2\ell}{2}\right)}, \quad (45)$$

$$\frac{\partial\mathcal{B}}{\partial a} = -d\sum_{\ell=2} R^{\ell}\mathcal{G}_{\ell}\left(\frac{3+2\ell}{2}\right)\left(\frac{5+2\ell}{2}\right) a^{-\left(\frac{7+2\ell}{2}\right)}, \quad (46)$$

$$\frac{\partial\mathcal{C}}{\partial a} = -\sum_{\ell=2} R^{\ell}\mathcal{G}_{\ell}\left(\frac{3+2\ell}{2}\right) a^{-\left(\frac{5+2\ell}{2}\right)}, \quad (47)$$

$$\frac{\partial\mathcal{B}}{\partial d} = \sum_{\ell=2} R^{\ell}\mathcal{G}_{\ell}\left(\frac{3+2\ell}{2}\right) a^{-\left(\frac{5+2\ell}{2}\right)}. \quad (48)$$

For $\ell = 2$ and a GPS orbit, by assuming $a = 26578$ km, $i = 55$ deg and $d = 5$ km the errors in $\Delta\xi$ are[8]

$$[\delta(\Delta\xi)]_{\Delta\dot{\Omega}} = \frac{\delta(\Delta\dot{\Omega})}{\sqrt{GM}(\mathcal{C}-\mathcal{B})} = 2\times 10^{-8}\ (\text{mas/yr})^{-1}\times\delta(\Delta\dot{\Omega}), \quad (49)$$

[7]Here we neglect the non–gravitational perturbations on the nodes. Contrary to, e.g., the perigees ω, the nodes are rather insensitive to such non–geodesic accelerations (Lucchesi 2001; 2002). In regard to the Earth solid tides, they have been neglected because their impact is several orders of magnitude smaller. For example, the amplitude of the nodal rate perturbation induced by the K_1 tide is five orders of magnitude smaller than that due to the even zonal harmonic coefficient J_2 of the geopotential for a GPS orbit.

[8]In the calculations it turns out that the effect of $\Delta\mathcal{G}_2$ in $\mathcal{A}$ and $\frac{\partial\mathcal{A}}{\partial a}$, for $\Delta i = 1$ deg, can be neglected. On the other hand, with a good quality rocket launcher it is possible to insert two spacecrafts in the same orbital planes up to 10^{-4} deg, as in the case of the GRACE mission. See on the WEB http://www.csr.utexas.edu/grace/newsletter/2002/august2002.html

$$[\delta(\Delta\xi)]_{GM} = \frac{(\Delta\dot{\Omega})_{\rm exp}}{2\sqrt{(GM)^3}(\mathcal{C}-\mathcal{B})}\times\delta(GM) = 3\times10^{-20}\ ({\rm deg/day})^{-1}\times(\Delta\dot{\Omega})_{\rm exp}, \tag{50}$$

$$[\delta(\Delta\xi)]_{J_2} = \left|\left\{\frac{\frac{\partial(\mathcal{A}+\mathcal{B})}{\partial J_2}\times(\mathcal{C}-\mathcal{B})-\frac{\partial(\mathcal{C}-\mathcal{B})}{\partial J_2}\times(\mathcal{A}+\mathcal{B})}{(\mathcal{C}-\mathcal{B})^2}\right\}\right|\times\delta J_2 = 2\times10^{-9}, \tag{51}$$

$$[\delta(\Delta\xi)]_{a} = \left|\left\{\frac{\frac{\partial(\mathcal{A}+\mathcal{B})}{\partial a}\times(\mathcal{C}-\mathcal{B})-\frac{\partial(\mathcal{C}-\mathcal{B})}{\partial a}\times(\mathcal{A}+\mathcal{B})}{(\mathcal{C}-\mathcal{B})^2}\right\}\right|\times\delta a = 2\times10^{-13}\ {\rm cm}^{-1}\times\delta a, \tag{52}$$

$$[\delta(\Delta\xi)]_{d} = \left|\left\{\frac{\frac{\partial\mathcal{B}}{\partial d}\times(\mathcal{C}+\mathcal{A})}{(\mathcal{C}-\mathcal{B})^2}\right\}\right|\times\delta d = 1\times10^{-9}\ {\rm cm}^{-1}\times\delta d. \tag{53}$$

It can be noticed that the major limiting factor is the term due to the error in the difference of the nodal rates $\delta(\Delta\dot{\Omega}) = \delta\dot{\Omega}^{(1)} + \delta\dot{\Omega}^{(2)}$. Indeed, the experimental error in measuring the secular rate of the node is of the order of 1 mas yr^{-1}. The systematic error in $\dot{\Omega}$ due to the uncertainty on J_2 is, for a GPS satellite, almost 3 mas yr^{-1}. In the case of the error in d, the same considerations as for the orbital periods hold.

4 Conclusions

In this paper we have shown that the comparison of the orbital motions of a pair of artificial satellites of different compositions moving along identical orbits in the gravitational field of Earth in order to test the Weak Equivalence Principle is not competitive with the already performed tests with torsion balances on Earth and the Lunar Laser Ranging technique, and the dedicated space–based missions STEP, GG, μSCOPE and SEE.

We have considered the orbital periods and the secular nodal rates. The analysis of the orbital periods seems to yield more precise measurements. The major limiting factor is represented by the difference in the orbital radiuses induced by the unavoidable orbital injection errors and the related uncertainty. By assuming $\delta d \leq 1$ cm or less the achievable precision is of the order of 10^{-10}–10^{-11}.

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